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Burgers turbulence model for a channel flow

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The truncated Burgers models have a unique equilibrium state which is defined continuously for all the Reynolds numbers and attainable from a realizable class of initial disturbances. Hence, they represent a sequence of convergent approximations to the original (untruncated) Burgers problem. We have pointed out that consideration of certain degenerate equilibrium states can lead to the successive turbulence–turbulence transitions and finite-jump transitions that were suggested by Case & Chiu. As a prototype of the Navier–Stokes equations, Burgers model can simulate the initial-value type of numerical integration of the Fourier amplitude equations for a turbulent channel flow. Thus, the Burgers model dynamics display certain idiosyncrasies of the actual channel flow problem described by a truncated set of Fourier amplitude equations, which includes only a modest number of modes due to the limited capability of the computer at hand.

1. Introduction and summary

Recently, Case & Chiu (1969) have investigated the stability of several of the lowest-order truncated Burgers (1937, 1948) models for a turbulent channel flow, which were obtained by discarding all the disturbance modes of order higher than a certain preassigned one. Following their terminology, let us denote the laminar solution by L and the tubulent solutions by $T_{(m)}$, where m is the number of nodes within the channel width. Here, for the lack of appropriate name, we shall loosely call all solutions of the Burgers model other than the L, turbulent solutions. The naturally expected stability result is the existence of the well-known laminar transition of $L \rightarrow T_{(0)}$ which occurs at a certain critical Reynolds number, R_c, for all the truncated models. This laminar-turbulence (L–T) transition takes place not only continuously but reversibly with respect to the Reynolds number, R. What is not anticipated, however, from the usual stability analysis is the surprising result that the truncated Burgers models can further induce a series of the turbulence–turbulence (T–T) transitions; i.e. $T_{(0)} \rightarrow T_{(1)}$ at the second R_c, $T_{(1)} \rightarrow T_{(2)}$ at the third R_c, and so on. Furthermore, a thought-provoking conclusion of their perturbation analysis is the possibility of a so-called finite-jump transition, whereby certain T–T transitions involve a discontinuous change in the turbulent solutions. If their stability conclusions were true, then the equilibrium turbulent solutions of a truncated model will not in general be continuous with respect to R. Further, as the truncation order is increased, the equilibrium solution of the corresponding Burgers model will involve increasingly many different

$T_{(m)}$ which are adjointed discontinuously at some R_c. Hence, there is no hope for approximating the original (untruncated) Burgers problem by the truncated models in any convergent fashion.

Our general stability conclusion is this: the truncated Burgers models have only the L–T transition of $L \rightarrow T_{(0)}$ at the critical $R_c = \pi$ and hence the T–T transitions of the kind that Case & Chiu have proposed are not realizable in actuality. In the range $R < R_c$, the laminar solution L is asymptotically stable and the region of asymptotic stability includes the entire phase-space by the Zubov (1964) theorem. Hence, L is attainable from any initial data. In the turbulent range $R > R_c$, however, the truncated Burgers models have a unique equilibrium state defined continuously for all R. This equilibrium state is not only stable in the sense of Lyapunov (see Hahn 1963), but also attainable by directly integrating the model equations from a set of class A initial data. Here, we bisect the initial data space by including first in class A only the points whose primary mode has an amplitude larger than that of any other mode and in class B all those not in class A. It has been found that the truncated Burgers models have a convergent sequence of turbulent equilibrium states which are attainable from the class A initial data. On the other hand, the truncated Burgers models under the class B initial data lead to degenerate equilibrium states which show no convergence property with respect to the truncation approximation. It must be pointed out that the separation of the initial data space is empirical and based solely on the convergence of truncated Burgers models. It however reflects a quirk of the Burgers model in that for the Navier–Stokes equations the class B initial data are just as meaningful as the class A. In addition, the truncated Burgers models have another kind of degenerate equilibrium state which cannot be attained from any initial data, perhaps, except from the equilibrium state itself. The most peculiar feature of the Burgers model is that the number of nodes for $T_{(m)}$ is not determined uniquely by the problem, although its maximum number is dictated by the Reynolds number. It is this arbitrariness in m that has led Case & Chiu (1969) to suggest the successive T–T transitions. Since m is an external parameter, the truncated Burgers models do not in actuality induce a series of T–T transitions as we increase R steadily. It will be shown in § 3·1 that the T–T and finite–jump transitions have been predicted in an attempt to match up certain equilibrium states of degenerate kind.

Although the stability analysis of truncated Burgers models is important in its own right, the ultimate purpose of this paper is to simulate the initial-value type of numerical integration of the Navier–Stokes equations for a turbulent channel flow. Let us consider a steady channel flow between two parallel infinite plates under a large external pressure gradient. If the flow field is Fourier analyzed in the longitudinal plane of homogeneity and if the flow variables are discretized along the channel width, then the Navier–Stokes equations have a representation of Fourier amplitude equations which are an infinite set of ordinary differential equations. Due to the inherent complexity and strong coupling among the Fourier modes, it is not possible to carry out numerical integration of the Fourier amplitude equations for both a large number of modes and a long time-period using the computer at present available to the author. Since the Burgers model shares the same basic non-linear structure with the Navier–Stokes equations, it is hoped that certain behaviours of the Fourier amplitude equations for a channel flow can be explained from the Burgers model which can readily be investigated in depth. First of all, it is not only justified but advantageous to use the so-called quasi-steady formulation (§§ 3·2 and 4) in which the initial-value type of integration has a physical analogue of following the growth of an initial disturbance introduced into the otherwise quasi-steady laminar flow. Secondly, due to the limited storage of a computer, we are forced to truncate the Fourier amplitude equations at a rather low level. As exhibited by the Burgers model, the truncation approximation tends to bring about an excessive transfer of the mean motion energy to the disturbances. Lastly, since the Burgers model maps the class A initial data into a unique equilibrium state, it is evident that this equilibrium state also represents the stationary statistical dynamics provided the initial ensemble is chosen from the class A initial data. This is the most interesting property of the Burgers model shared by the Navier–Stokes equations which have the universal stationary distributions of the mean flow and root-mean-square of the turbulent velocities in a channel.

2. Burgers model equations

In order to enhance the analogy between the Burgers model and actual turbulent channel flow problem, we shall write the basic equations of Burgers in the dimensionless form which closely resemble the Navier–Stokes equations for a channel flow

$$dU/dt = 1 - U/R - \int_0^1 v^2\,dy, \tag{2.1}$$

$$\partial v/\partial t = Uv + R^{-1}\,\partial^2 v/\partial y^2 - \partial v^2/\partial y. \tag{2.2}$$

Here, we may consider $R = u^* h/\nu$ as a Reynolds number based on the friction velocity $u^* = P^{\frac{1}{2}}$ (P being the external pressure), the channel width h, and the kinematic viscosity ν. Let us call (2.1) and (2.2) respectively the mean and disturbance equations, thereby gaining access to shear flow terminology. Under the no-slip wall velocities $v(0) = v(1) = 0$, the total energy of the flow system obeys the following energy balance equation obtained from (2.1) and (2.2) in the usual manner

$$\frac{1}{2}\frac{d}{dt}\left(U^2 + \int_0^1 v^2\,dy\right) = U - R^{-1}\left[U^2 + \int_0^1 \left(\frac{\partial v}{\partial y}\right)^2 dy\right]. \tag{2.3}$$

The physical interpretation of (2.3) is that the temporal variation of the total energy is due to the balance between the energy input by the external pressure force and the energy loss by viscous dissipation.

2.1. *Burgers solution for the case of $U = R$*

The steady mean motion can be found immediately from (2.1)

$$U = R\left[1 - \int_0^1 v^2\,dy\right]. \tag{2.4}$$

This has the familiar form of the laminar flow $U = R$ as modified by the Reynolds shear stress which provides the effective mean-disturbance interaction. For the special case of no Reynolds shear, Burgers (1937) has obtained the stationary solution of (2.2) with a constant U in the following parametric form

$$\left.\begin{aligned} v &= \pm[U/2R]^{\frac{1}{2}}[C-\eta+\ln(1+\eta)]^{\frac{1}{2}}, \\ y &= \left[\frac{1}{2RU}\right]^{\frac{1}{2}}\int_{\eta_1}^{\eta_2}\frac{d\eta}{(1+\eta)[C-\eta+\ln(1+\eta)]^{\frac{1}{2}}}. \end{aligned}\right\} \tag{2.5}$$

Here, the auxiliary variable $\eta = -2(\partial v/\partial y)/U$ must lie between the two limits η_1 and η_2, which are the distinct roots of $C-\eta+\ln(1+\eta) = 0$. The peculiar feature of (2.5) is that the integration constant C cannot be chosen arbitrarily and it must take on certain discrete values, similar to an eigenvalue. Since $v = 0$ at $\eta = \eta_1$, we associate the lower limit η_1 with $y = 0$. We then observe that v is again zero at $\eta = \eta_2$. Hence, by judicious selection of C it is possible to introduce an arbitrary number of nodes for v in $(0, 1)$. Namely, we can find for $T_{(0)}$ a constant C_0 such that (2.5) gives $y = 1$, for $T_{(1)}$ another constant C_1 such that (2.5) gives $y = \frac{1}{2}$, and so forth. Thus, the condition that v has m nodes becomes

$$\left[\frac{1}{2UR}\right]^{\frac{1}{2}}\int_{\eta_1}^{\eta_2}\frac{d\eta}{(1+\eta)[C_m-\eta+\ln(1+\eta)]^{\frac{1}{2}}} = \frac{1}{m+1},$$

provided C_m can be found. Burgers has pointed out that the sequence of C_m is decreasing because the definite integral decreases with C and further that the definite integral has its minimum $\pi 2^{\frac{1}{2}}$ as $C_m \to 0$. Since we have $1/(m+1) = \pi/R$ as $C_m \to 0$, the maximum number of nodes for a fixed R is restricted to the largest integer contained in $[R/\pi - 1]$. Otherwise, the choice of m is completely arbitrary as long as it does not exceed the maximum limit. This, however, should not be interpreted as the emergence of new nodes as R passes through multiples of π. The obvious reason is that m is an external parameter and hence, for instance, $T_{(0)}$ can exist for all $R > \pi$ without making a transition to $T_{(1)}$.

2.2. *The infinite system of amplitude equations*

For the general case of $U \neq R$, we may expand the velocity field in a complete set of the basis functions satisfying the boundary conditions

$$v(y,t) = \sum_{n=1}^{\infty}\xi_n(t)\sin(n\pi y). \tag{2.6}$$

Introducing (2·6) into (2.1) and (2.2), the mean and disturbance equations become

$$dU/dt = 1 - U/R - \frac{1}{2}\sum_{n=1}^{\infty}\xi_n^2, \tag{2.7}$$

$$d\xi_n/dt = [U-(n\pi)^2/R]\,\xi_n - n\pi\left[\frac{1}{2}\sum_{k=1}^{n-1}\xi_k\,\xi_{n-k} - \sum_{k=1}^{\infty}\xi_k\,\xi_{n+k}\right]. \tag{2.8}$$

Considering U as the zeroth harmonic, we see that (2.7) and (2.8) represent an infinite set of amplitude equations. The non-linear dynamics are expressed by the convolution sums in the right-hand side of (2.8). Physically speaking, the first sum $\frac{1}{2}\Sigma\xi_k\xi_{n-k}$ represents the absorption of energy from the modes of order lower than ξ_n, whereas the second sum $-\Sigma\xi_k\xi_{n+k}$ is responsible for the transmission of energy to all the modes of order higher than ξ_n. Naturally, the first convolution sum is absent in the ξ_1 equation.

An interesting observation can be made from the total energy balance equation (2.3) which now has the form

$$\frac{1}{2}\frac{d}{dt}\left[U^2+\frac{1}{2}\sum_{n=1}^{\infty}\xi_n^2\right] = U-\frac{1}{R}\left[U^2+\frac{1}{2}\sum_{n=1}^{\infty}(n\pi)^2\xi_n^2\right]. \tag{2.9}$$

If there is an equilibrium state, then since the left-hand side of (2.9) is expected to vanish identically, the equilibrium trajectories of U and ξ_n are constrained by the ellipsoid

$$(U-\tfrac{1}{2}R)^2+\frac{1}{2}\sum_{n=1}^{\infty}(n\pi)^2\xi_n^2 = (\tfrac{1}{2}R)^2.$$

In fact, the ellipsoid is an invariant set. In order to describe the full transient dynamics, we must treat the set of (2.7) and (2.8) for the Burgers model problem. However, when the investigation of equilibrium states is at issue, it is indeed expedient to examine a degenerate system of (2.8) and (2.4), the latter of which now has the following representation:

$$U = R\left[1-\frac{1}{2}\sum_{n=1}^{\infty}\xi_n^2\right]. \tag{2.10}$$

The set of (2.8) and (2.10) will be called hereafter the quasi-steady system. The advantage of the quasi-steady formulation will later be discussed in §3.2 by comparing the phase-space behaviour of the quasi-steady system with that of the original system.

2.3. *Stability analysis*

In the quasi-steady formulation, it is natural to visualize the disturbances ξ_n to represent a certain perturbation superimposed on the otherwise laminar mean motion $U = R$. Physically speaking, we expect that a disturbance of any kind will damp out in the small R range, thereby recovering the laminar motion regardless of the initial excitation. On the other hand, it is plausible to anticipate that as $R > R_c$ a certain class of the initial disturbances may develop into an equilibrium state which can be identified as the turbulent flow. To put this physical concept in a firm mathematical framework, we rewrite (2.8) in the vector form

$$d\mathbf{x}/dt = D\mathbf{x}+F(\mathbf{x}) \equiv G(\mathbf{x}), \tag{2.11}$$

where $\mathbf{x}$ is a column vector with the components ξ_n, D is a diagonal matrix with the components $D_{nn} = U-(n\pi)^2/R$, and $F(\mathbf{x})$ is a column vector with the non-linear components

$$\left\{-n\pi\left[\frac{1}{2}\sum_{k=1}^{n-1}\xi_k\xi_{n-k}-\sum_{k=1}^{\infty}\xi_k\xi_{n+k}\right]\right\}.$$

It is important to observe that the energy conservation by the non-linear terms gives rise to the orthogonality between $\mathbf{x}$ and $F(\mathbf{x})$, i.e.

$$\mathbf{x}^TF(\mathbf{x}) = 0, \tag{2.12}$$

where $\mathbf{x}^T$ is the transpose of $\mathbf{x}$. We further note that $F(\mathbf{x})$ satisfies the usual non-linearity condition (Bellman 1953) $\|F(\mathbf{x})\|/\|\mathbf{x}\| \to 0$ as $\|\mathbf{x}\| \to 0$, where $\|\;\|$ is a metric norm. Therefore, it is obvious that $\mathbf{x} = 0$ is the null solution of (2.11). According to the fundamental stability theorem (Bellman 1953), $\mathbf{x} = 0$ is a stable solution of (2.11) if D is a stable matrix, i.e. all the eigenvalues of D have negative real parts. In fact, D is stable in the laminar range $R < \pi$, and hence $\mathbf{x} = 0$ represents the laminar solution.

In the turbulent range $R > \pi$, some of the eigenvalues of D become positive. Hence, the fundamental stability theorem based on the linear behaviour of (2.11) can no longer be applied. Although some components of $\mathbf{x}$ will then be amplified according to linearized stability theory, it is quite conceivable to expect that their growth may eventually be modulated by the non-linear terms so as to attain a stable equilibrium state different from $\mathbf{x} = 0$. In order to encompass non-linearity in the stability investigation of (2.11), we introduce a Lyapunov function (Hahn 1963)

$$V = \tfrac{1}{2}\mathbf{x}^T\mathbf{x},$$

which is clearly positive definite, since it represents the disturbance energy. The stability criteria of Lyapunov theory are based on definiteness of the total time derivative of V which can be computed through (2.11) (in view of (2.12))

$$dV/dt \equiv \dot{V} = \mathbf{x}^T D\mathbf{x}. \tag{2.13}$$

In the laminar range, we note that $\dot{V} < 0$ because D is a stable matrix. Hence, $\mathbf{x} = 0$ is asymptotically stable. Zubov (1964) has suggested a method for finding the region of asymptotic stability of the null solution from a Lyapunov function V_1 which is the solution of the following partial differential equation

$$[dV_1(\mathbf{x})/dt]_{t=0} = \phi(\mathbf{x})\,[1+G^TG]^{\frac{1}{2}}\,(1+V_1(\mathbf{x})). \tag{2.14}$$

With the choice of a positive definite function $\phi(\mathbf{x}) = -\mathbf{x}^TD\mathbf{x}/[1+G^TG]^{\frac{1}{2}}$, the Lyapunov function which satisfies (2.14) and all the conditions of Zubov (1964, theorem 22) is $V_1 = \exp(-\frac{1}{2}\mathbf{x}^T\mathbf{x}) - 1$. Since the region of asymptotic stability is bounded by $V_1 = -1$, we see that it is the entire $\mathbf{x}$ space and hence the null solution is completely stable.

In the turbulent range, the equilibrium state $\mathbf{x} = \boldsymbol{\xi}$ is the solution of

$$D\mathbf{x} + F(\mathbf{x}) = 0. \tag{2.15}$$

In view of the orthogonality (2.12), we can derive from (2.15) a very important condition on the equilibrium states

$$\boldsymbol{\xi}^T D\boldsymbol{\xi} = 0. \tag{2.16}$$

The immediate consequence of (2.16) is that we can conclude from (2.13)

$$\dot{V} = 0. \tag{2.17}$$

This is a sufficient condition for the equilibrium state $\boldsymbol{\xi}$ to be stable. Furthermore, we can derive from (2.16) certain conditions for (2.15) to have a real equilibrium state. Clearly, at least one eigenvalue of D must be positive, otherwise (2.16) cannot be satisfied by non-zero $\boldsymbol{\xi}$. Some other conditions imposed by (2.16) will later be encountered in § 3.1.

Alternatively, the local stability of (2.11) about an equilibrium state can be examined by investigating stability of the Jacobian matrix of G with respect to $\mathbf{x}$ at $\mathbf{x} = \boldsymbol{\xi}$.

$$J(G,\mathbf{x})_{\mathbf{x}=\bar{\xi}} = \begin{bmatrix} (\bar{U}-\pi^2/R)+\pi\bar{\xi}_2 & \pi(\bar{\xi}_1+\bar{\xi}_3) & \pi(\bar{\xi}_2+\bar{\xi}_4) & - \\ 2\pi(-\bar{\xi}_1+\bar{\xi}_3) & (\bar{U}-(2\pi)^2/R)+2\pi\bar{\xi}_4 & 2\pi(\bar{\xi}_1+\bar{\xi}_5) & - \\ 3\pi(-\bar{\xi}_2+\bar{\xi}_4) & 3\pi(-\bar{\xi}_1+\bar{\xi}_5) & (\bar{U}-(3\pi)^2/R)+3\pi\bar{\xi}_6 & - \\ - & - & - & - \end{bmatrix}$$

where $\bar{U}$ is given by (2.10) in which ξ_n are replaced by $\bar{\xi}_n$. Whether or not $J(G,\mathbf{x})$ is stable will be determined by the Hurwitz conditions.

3. The truncated set of the quasi-steady system

To simplify the coefficients of the amplitude equations, let us introduce into (2.8) and (2.10)

$$U = R\hat{U}, \quad R = \pi\hat{R}, \quad t = \hat{t}/\pi, \tag{3.1}$$

and remove the $\wedge$'s in the resulting equations

$$\left.\begin{aligned} U &= 1-\frac{1}{2}\sum_{n=1}^{\infty}\xi_n^2, \\ d\xi_n/dt &= R(U-(n/R)^2)\,\xi_n - n\left[\frac{1}{2}\sum_{k=1}^{n-1}\xi_k\,\xi_{n-k} - \sum_{k=1}^{\infty}\xi_k\,\xi_{n+k}\right]. \end{aligned}\right\} \tag{3.2}$$

The new R is now measured in units of π. Since we are measuring U in units of R, the new U is normalized. The amplitudes of ξ_n are not at all affected by (3.1), however. Clearly, $T_{(0)}$ is governed by the set of (3.2) in its entirety. On the other hand, $T_{(m)}$ for $m \geqslant 1$ is the solution of the reduced set of (3.2) in which are retained only the ξ's whose harmonic indices are multiples of $(m+1)$. Therefore, we see at once that the $T_{(0)}$ system is identical to any one of the $T_{(m)}$ systems if all the harmonic indices of the $T_{(0)}$ system are multiplied by the factor $(m+1)$. Since the $T_{(m)}$ systems (for all admissible m) form a class of equivalent dynamic problems, it suffices to investigate only the $T_{(0)}$ system as the investigation of any other $T_{(m)}$ system would certainly be redundant. This is expected because we have constructed the $T_{(m)}$ by arranging $(m+1)$ of $T_{(0)}$ in such a way that continuity is preserved at the nodes.

Although the $T_{(0)}$ system is formally equivalent to the original equations (2.2) and (2.4), the expansion technique has produced an infinite set of amplitude equations which is theoretically intractable. Hence, we shall truncate it by discarding all the disturbance modes of order higher than those we wish to retain. Of course, convergence is the basic premise for the truncation approximation which assures that the accuracy of approximation can be improved by including arbitrary large number of modes. The main objective of this section is to demonstrate the equilibrium state dynamics of the truncated Burgers models, and hence we shall investigate in detail the first three truncated sets by considering them as models for a certain dynamic process. Prior to the detailed discussion, we must point out that the truncated Burgers model has certain degenerate turbulent solutions. Let us denote the truncated set of (3.2) by $T^{(p)}_{(0)}$, where p is the number of modes retained. Suppose that we excite initially only the mode, say, k whose second multiple $2k$ exceeds the upper limit mode p. Then, the truncated Burgers model cannot display the desired non-linear dynamics of dispersing the initial single mode energy to its neighbouring Fourier modes. In particular, under the initial condition

$$\left.\begin{aligned} \xi_k(0) &= c \quad (2k > p, c \text{ being an arbitrary constant}),\\ \xi_j(0) &= 0 \quad (j \neq k, j \leqslant p), \end{aligned}\right\} \tag{3.3}$$

the equilibrium turbulent solution of $T^{(p)}_{(0)}$ for $R > k$ is seen to be

$$\bar{U} = (k/R)^2, \quad \bar{\xi}^2_k = 2(1-(k/R)^2), \quad \bar{\xi}_j = 0 \quad (j \neq k). \tag{3.4}$$

We shall hereafter call (3.4) the degenerate turbulent solution, for it represents the frozen state of motion at a single Fourier mode. Since (3.3) has zero measure in the initial data space, we shall in general consider (3.4) to represent a trivial state unless it can be attained from some other initial conditions.

3.1. *Several truncated sets of $T_{(0)}$ system*

The lowest-order truncated set of (3.2) is $T^{(1)}_{(0)}$ which Case & Chiu have discounted as a trivial case.

The $T^{(1)}_{(0)}$ truncated set. We have for $p = 1$

$$\left.\begin{aligned} U &= 1-\tfrac{1}{2}\xi_1^2,\\ d\xi_1/dt &= R(U-1/R^2)\,\xi_1. \end{aligned}\right\} \tag{3.5}$$

For $R < R_c = 1$, the equilibrium solution is L. In the turbulent range $R > R_c$, since (2.16) requires that $\bar{U} - 1/R^2 = 0$, the real equilibrium state is

$$\left.\begin{aligned} \bar{U} &= 1/R^2,\\ \bar{\xi}_1 &= \pm[2(1-1/R^2)]^{\frac{1}{2}}. \end{aligned}\right\} \tag{3.6}$$

The sign of $\bar{\xi}_1$ is determined by its initial sign. Although (3.6) coincides with (3.4) for $k = 1$, it is not a trivial equilibrium state because we have verified that (3.6) is attainable from the arbitrary initial data. In view of the asymptotic behaviour $\bar{U} \to 0$ and $\bar{\xi}_1 \to \pm 2^{\frac{1}{2}}$ as $R \to \infty$, the truncated set (3.5) represents a physical process of transferring the mean motion energy to the disturbance. As we shall see later, an excessive mean-to-disturbance energy transfer is the common feature shared by all truncated Burgers models.

The $T^{(2)}_{(0)}$ truncated set. The next truncated set is the lowest-order case that Case & Chiu (1969) have examined

$$\left.\begin{aligned} U &= 1-\tfrac{1}{2}(\xi_1^2+\xi_2^2),\\ d\xi_1/dt &= R(U-1/R^2)\,\xi_1+\xi_1\xi_2,\\ d\xi_2/dt &= R(U-4/R^2)\,\xi_2-\xi_1^2. \end{aligned}\right\} \tag{3.7}$$

Again, L is the equilibrium solution for $R < R_c = 1$. In the range $R_c < R < (17/2)^{\frac{1}{2}}$ only the first eigenvalue of (2.16) is positive and hence the real equilibrium state of (3.7) becomes

$$\left.\begin{aligned} \overline{U} &= (2/5)\,[1+3/2R^2], \\ \overline{\xi}_1 &= \pm\frac{2R}{5}\left[-\left(1-\frac{17}{2R^2}\right)\left(1-\frac{1}{R^2}\right)\right]^{\frac{1}{2}}, \\ \overline{\xi}_2 &= -(2R/5)\,[1-1/R^2]. \end{aligned}\right\} \tag{3.8}$$

As in (3.6) the sign of $\overline{\xi}_1$ is dictated by its initial sign. For $R > (17/2)^{\frac{1}{2}}$, however, (2.16) cannot be satisfied by two positive eigenvalues, hence we have for the real equilibrium state

$$\left.\begin{aligned} \overline{U} &= 4/R^2, \\ \overline{\xi}_1 &= 0, \\ \overline{\xi}_2 &= -[2(1-4/R^2)]^{\frac{1}{2}}. \end{aligned}\right\} \tag{3.9}$$

It must be pointed out that (3.8) and (3.9) have the common value at $R = (17/2)^{\frac{1}{2}}$ and they represent a unique equilibrium state defined continuously for all R. Here, again, (3.9) which coincides with (3.4) for $k = 2$ is not trivial because it can be attained from the arbitrary initial data. Since (3.9) also represents the equilibrium solution of $T^{(1)}_{(1)}$, Case & Chiu have called $R = (17/2)^{\frac{1}{2}}$ the second R_c for the T–T transition of $T_{(0)} \to T_{(1)}$. Such an assertion, however, does not fit in the general convergence structure of the truncation approximation.

The $T^{(3)}_{(0)}$ *truncated set.* In contrast to the previous two truncated sets which had only one equilibrium state (except for the signs for $\overline{\xi}_1$), the present $T^{(3)}_{(0)}$ has several equilibrium states corresponding to the algebraic solutions of (2.15). As it turns out, all the equilibrium states are stable as they satisfy (2.17), hence we cannot choose *a priori* a correct equilibrium state into which $T^{(3)}_{(0)}$ will map a certain set of realizable initial data. The ambiguity will therefore be resolved by investigating attainability of each equilibrium state from the class A initial data. The closed equations for $T^{(3)}_{(0)}$ are

$$\left.\begin{aligned} U &= 1-\tfrac{1}{2}(\xi_1^2+\xi_2^2+\xi_3^2), \\ d\xi_1/dt &= R(U-1/R^2)\,\xi_1+\xi_1\xi_2+\xi_2\xi_3, \\ d\xi_2/dt &= R(U-4/R^2)\,\xi_2-\xi_1^2+2\xi_1\xi_3, \\ d\xi_3/dt &= R(U-9/R^2)\,\xi_3-3\,\xi_1\xi_2. \end{aligned}\right\} \tag{3.10}$$

Again, L is the laminar equilibrium solution. For $R > R_c$ the first eigenvalue of (2.16) is positive, hence a real equilibrium state of (3.10) becomes

$$\left.\begin{aligned} \overline{\xi}_1^2 &= \frac{R(\overline{U}-4/R^2)\ (\overline{U}-9/R^2)\,\overline{\xi}^2}{(\overline{U}-9/R^2)-6\overline{\xi}_2}, \\ \overline{\xi}_2 &= (R/6)\,[-(\overline{U}-9/R^2)-\{(\overline{U}-9/R^2)^2-12(\overline{U}-9/R^2)\,(\overline{U}-1/R^2)\}^{\frac{1}{2}}], \\ \overline{\xi}_3 &= 3\overline{\xi}_1\overline{\xi}_2/R(\overline{U}-9/R^2). \end{aligned}\right\} \tag{3.11}$$

We have chosen the minus sign for the discriminant of $\overline{\xi}_2$ in accordance with the

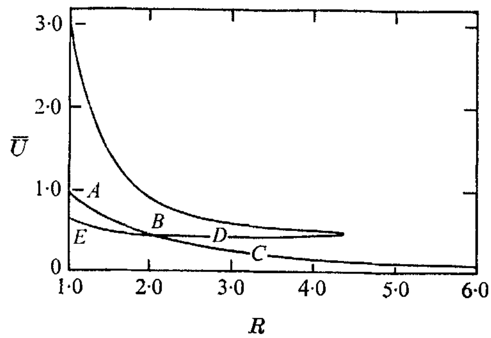

FIGURE 1. Equilibrium $\bar{U}$ versus R for $T^{(3)}_{(0)}$.

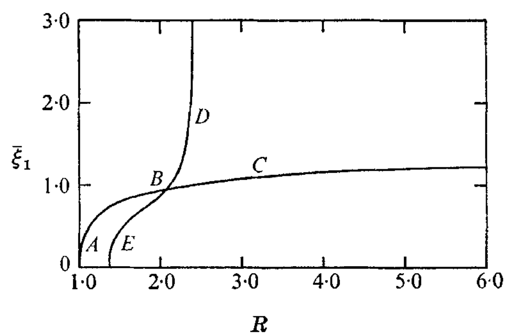

FIGURE 2. Equilibrium $\bar{\xi}_1$ versus R for $T^{(3)}_{(0)}$.

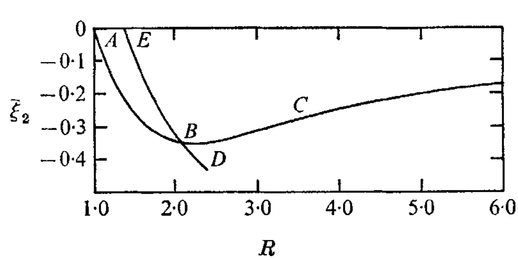

FIGURE 3. Equilibrium $\bar{\xi}_2$ versus R for $T^{(3)}_{(0)}$.

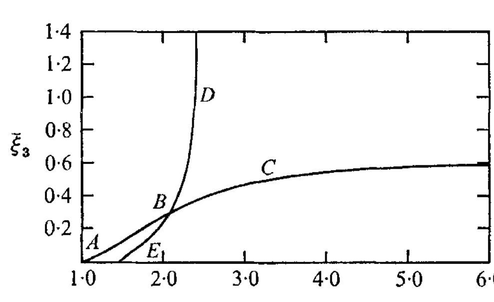

FIGURE 4. Equilibrium $\bar{\xi}_3$ versus R for $T^{(3)}_{(0)}$.

actual equilibrium solution. To complete the solution of (3.11) we obtain from the first of (3.10) a relation between $\overline{U}$ and R

$$\left(\overline{U}-\frac{9}{R^2}\right)\left[\left(24+\frac{27}{R^2}\right)-67\overline{U}\right]^2=\left(-11\overline{U}+\frac{3}{R^2}\right)\left[\left(12-\frac{3}{R^2}\right)-25\overline{U}\right]^2. \quad (3.12)$$

Although (3.11) and (3.12) are the same equations as used in Case & Chiu (1969) our present analysis of them gives an entirely different stability result.

We note that (3.12) is a cubic equation in $\overline{U}$ whose real roots are graphed in figure 1. Since $\overline{U} = 1$ at $R = 1$, the root locus A–B is the correct choice for small R and, by continuity, the relevant root locus is B–C for large R. For this $\overline{U}$ we find that (3.11) has a real equilibrium state which is depicted by the continuous curves A–B–C in the respective figures 2–4. Let us denote this equilibrium state A–B–C by $\bar{\xi}_{A-B-C}$. It has been verified that $\bar{\xi}_{A-B-C}$ is attainable from the class A initial data. Furthermore, $\bar{\xi}_{A-B-C}$ also reflects the excessive mean-to-disturbance energy transfer for large R. Suppose that, at the point B of figure 1, we had chosen the root locus B–D, then both $\bar{\xi}_1$ and $\bar{\xi}_3$ would approach the vertical asymptote at $R \simeq 2.43$. On the other hand, had we followed the root locus B–E, the solution of (3.11) becomes non-real for $R \lesssim 1.38$. In fact, the root locus B–E is not acceptable for the obvious reason that the laminar and turbulent solutions cannot be matched up continuously at $R = 1$. Although $\bar{\xi}_{E-B-D}$ satisfies (2.17) and the Jacobian matrix for it is stable, we have not been able to approach this equilibrium state from any initial data (other than $\bar{\xi}_{E-B-D}$ itself).

In a large R range, there is another equilibrium state

$$\left.\begin{aligned}\overline{U} &= 4/R^2,\\ \bar{\xi}_1 &= \bar{\xi}_3 = 0,\\ \bar{\xi}_2 &= \pm[2(1-4/R^2)]^{\frac{1}{2}}.\end{aligned}\right\} \quad (3.13)$$

As in the case of (3.9), (3.13) is not trivial because it is attainable from some initial data of class B other than (3.3). Case & Chiu have suggested the R range of (3.13) to be $R \gtrsim (5{\cdot}2)^{\frac{1}{2}}$, and we have confirmed this to be a correct estimate. Since (3.13) also represents the equilibrium solution of $T^{(1)}_{(1)}$, they have called $R \simeq (5{\cdot}2)^{\frac{1}{2}}$ the second R_c for the T–T transition of $T_{(0)} \rightarrow T_{(1)}$. They have further suggested this to be a finite-jump transition because there is no way that (3.13) can be matched continuously with either $\bar{\xi}_{B-C}$ or $\bar{\xi}_{B-D}$ at $R \simeq (5{\cdot}2)^{\frac{1}{2}}$. Theoretically in an even larger R range, (3.10) has another equilibrium state corresponding to (3.4) for $k = 3$. We have, however, not shown this to be a non-trivial state.

3.2. *Phase-space behaviours*

Thus far, our equilibrium state investigation has been based on the quasi-steady (Q–S) system (2.8) and (2.10) instead of the original system (2.7) and (2.8). The motivation for this was the belief that, if the original system ever attains an equilibrium state, it is the same state that Q–S system will attain because the existence of steady mean motion has been presupposed in the Q–S formulation. Although the Q–S system badly misrepresents the actual transient dynamics, it enjoys a tremendous computational advantage over the original system inasmuch as the search for equilibrium states is concerned. Figure 5 compares certain projections of the phase-space trajectory of the original system with the corresponding trajectory of the Q–S system under $R = 4$. In the first group of figures 5 (*a*)–(*c*), the trajectories of ξ's with respect to U are shown for the original system of $T^{(3)}_{(0)}$ under the initial data $\xi_1(0) = 0{\cdot}5$, $\xi_2(0) = 0$, $\xi_3(0) = 0{\cdot}1$, $U(0) = 1{\cdot}0$. We see that the trajectories go through three distinct stages: the exhaustive transfer of the mean motion energy to the disturbances, the back-flow of the accumulated disturbance energy to the mean motion, and the eventual energy equilibration toward a stationary state. The second group of figures 5 (*d*)–(*f*) show the trajectories of ξ's with respect to U for the Q–S system of $T^{(3)}_{(0)}$ under the

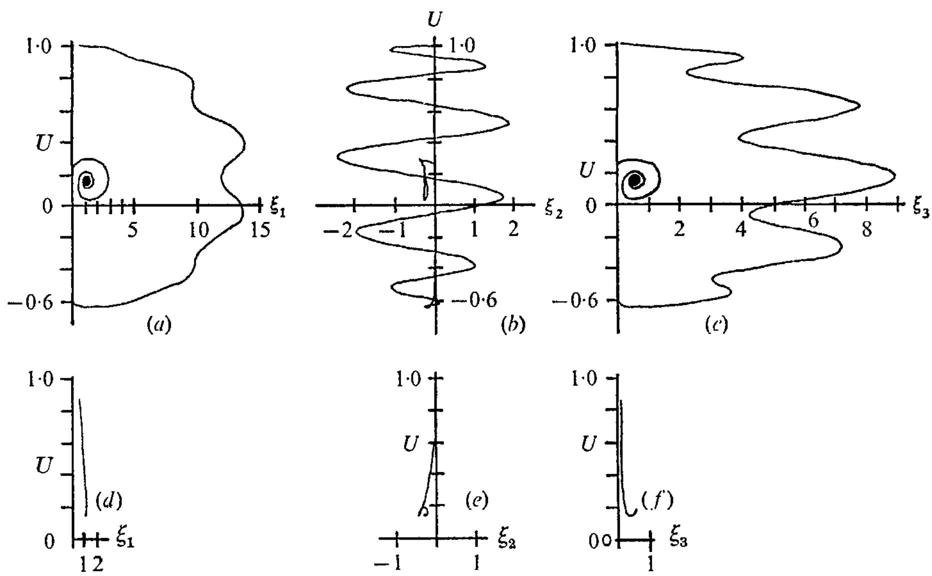

FIGURE 5. The phase-space trajectories ($R = 4{\cdot}0$). (*a*)–(*c*) The original system of $T^{(3)}_{(0)}$; (*d*)–(*f*) the quasi-steady system of $T^{(3)}_{(0)}$.

same initial data for $\boldsymbol{\xi}$ as in the first group of the figures. As anticipated, the Q–S system attains the same equilibrium state as the original system. Yet, the approach to equilibrium state in the phase-space representation is much more straightforward for the Q–S system than for the original system. However, figure 5 fails to indicate the actual evolution time required for equilibration. Indeed, the total equilibration time for the Q–S system is an order of magnitude shorter than that for the original system. All of these points are in favour of the Q–S system as an efficient means for the equilibrium state investigation.

4. The overall equilibrium state investigation

For the first three truncated sets of § 3.1, it was possible to enumerate all the equilibrium states of $T^{(p)}_{(0)}$ for $p = 1, 2, 3$ as the algebraic solutions of (2.15). The unique equilibrium state was then chosen based on the criteria that it is continuous in R and attainable from the class A initial data. Because of the analytical difficulty in solving (2.15) for a high-order truncation, it is not in general feasible to find algebraically the equilibrium states of $T^{(p)}_{(0)}$ for $p \geqslant 4$. Therefore, by reverting the previous procedure of picking out a unique equilibrium state among all possible ones, we shall resort to the direct search for a particular equilibrium state from the stationary solutions which are attainable from certain initial data. Of all the stationary solutions, we shall choose one for the equilibrium state that

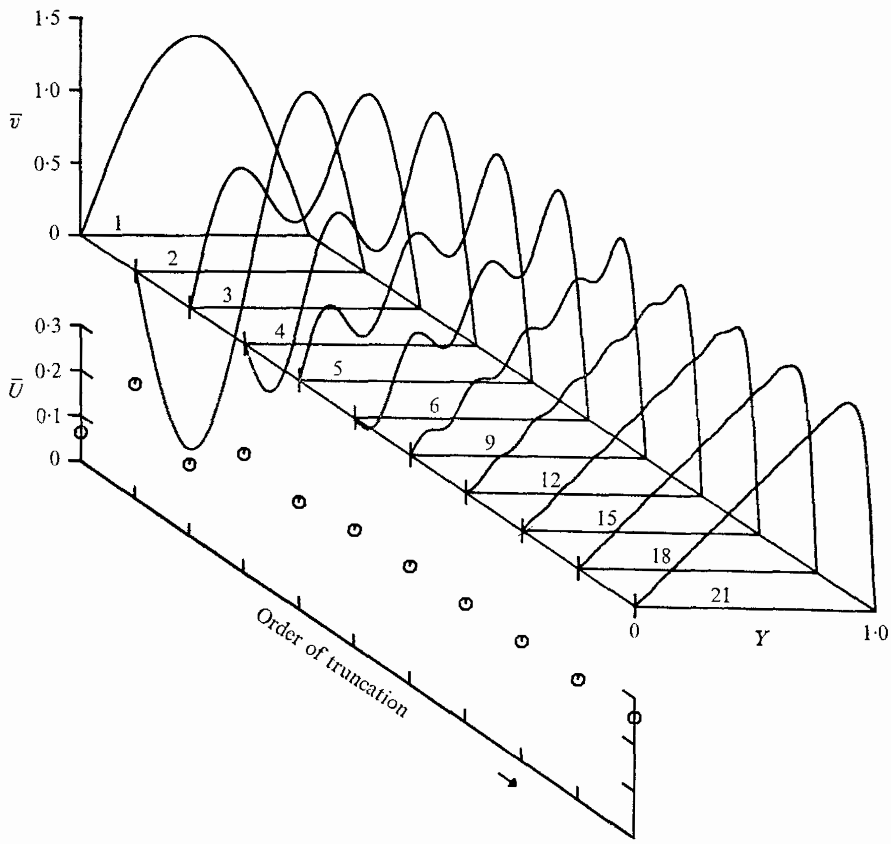

FIGURE 6. Equilibrium states of the truncated Burgers sets (R = 4·0).

satisfies the convergence requirement with respect to the truncation approximation. It has been found that the truncated sets $T^{(p)}_{(0)}$ for $p \geqslant 4$ have a unique equilibrium state which corresponds to the stationary solution attainable from the class A initial data. Since the search for equilibrium states has been carried out for discrete values of R, the continuity of equilibrium state with respect to R can only be asserted pointwise. In the way that the initial data space has been bisected, the equilibrium states attainable from the class B do not have the proper convergence property with respect to the truncation approximation. The entirety of such equilibrium states will not be discussed here, however.

We have summarized in figure 6 a typical sequence of the equilibrium states for the first 21 truncated sets which are all attainable from the class A initial data.

For a fixed R, we have chosen the mean motion $\overline{U}$ and the disturbance $\overline{v}$ (equation (2.6)) as the equilibrium state variables. The $\overline{U}$ and $\overline{v}$ of the first three truncated sets have already been presented in § 3.1. Let us make a few important observations from the figure. First of all, for a modest value of $R = 4$, it requires about $p = 21$ modes to suppress the truncation error to be $|\xi_i/\xi_1| < 0{\cdot}3\,\%$ $(i \geqslant 21)$. Clearly, as R increases it is necessary to include a larger number of modes in $T^{(p)}_{(0)}$ to maintain the same level of truncation error. Secondly, the lower-order truncated sets $(p \leqslant 6)$ give poor approximations to the original (untruncated) Burgers problem. Hence, their equilibrium dynamics should be interpreted only qualitatively. In particular, the $T^{(2)}_{(0)}$ has a typical solution of $T_{(1)}$. However, it is not consistent to associate such an irregularity with the T–T transition of $T_{(0)} \rightarrow T_{(1)}$.

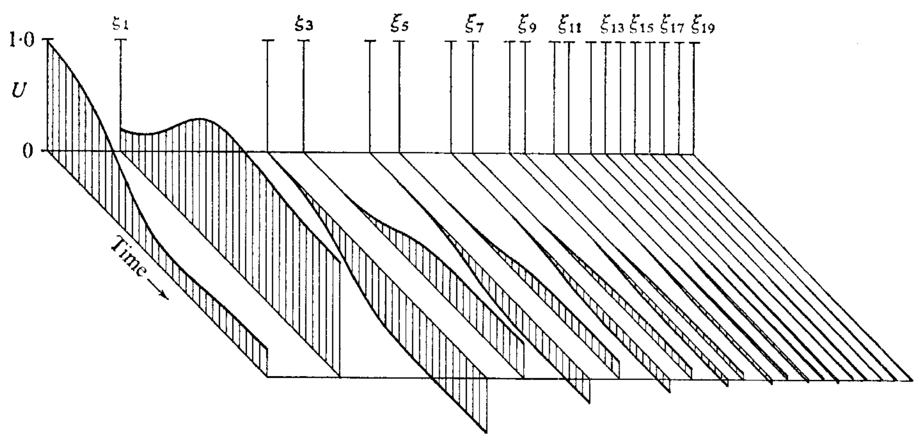

FIGURE 7. Development of the disturbance modes ($R = 4{\cdot}0$).

Apart from the phase-space behaviours already discussed in § 3.2, the quasi-steady formulation has an interesting physical motivation. Let a channel flow be sustained laminarly at an $R > R_c$. Suppose that we introduce into it a small but arbitrary perturbation of ξ_1 at the initial time. Then, two things will happen immediately. One is the growth of ξ_1 due to the influx of the mean motion energy by the mean-disturbance interaction, and the other is the excitation of the higher modes $\xi_2, \xi_3, \ldots$ due to the cascade energy flow by the non-linear interaction. As a whole, the flow of the mean motion energy to the disturbances is not an open-end process. This is because the Reynolds shear which modifies the laminar U has the effect of delimiting the level of mean energy drain by the mean-disturbance interaction. With the increase of ξ_1 amplitude, there will be more and more of the disturbance modes being excited by the non-linear interaction. Since the viscous dissipation takes place progressively with the higher modes, we can in actuality suppose the existence of a cut-off harmonic mode for a given R. Thereby, stationary turbulent channel flow can be established for a $T^{(p)}_{(0)}$. Figure 7 displays the development of a stationary channel flow in terms of the U and individual modes ξ_n which have been evolved from only the non-zero primary mode $\xi_1(0) = 0{\cdot}2$ and $\xi_i(0) = 0$ for $i \geqslant 2$. On the other hand, the development of the disturbance v under the same condition is shown in figure 8.

Before closing, we mention that both the digital and analog computers have been used in this work, capitalizing the best features of the two. Needless to say, an analog computer is most efficient in the search for equilibrium states, whereas the digital computer provides quantitative results. The numerical integration scheme used here was the one which was found practical, though not necessarily the most accurate, for the actual channel flow computation.

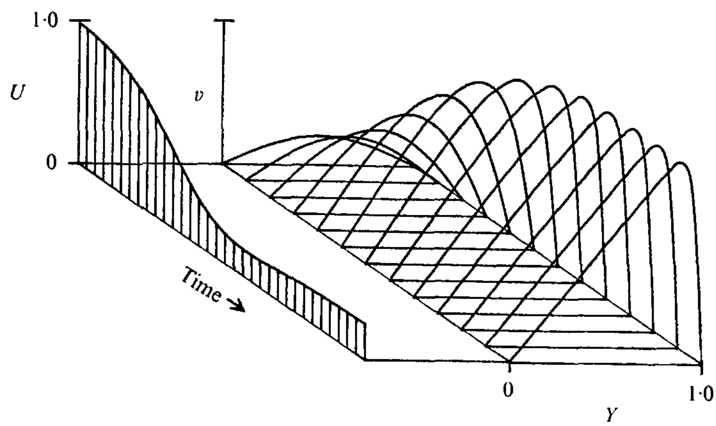

FIGURE 8. Development of the disturbance profile ($R = 4{\cdot}0$).

Returning to (2.11), we can obtain the following expression for $\mathbf{x}(t+\Delta t)$ (Δt being a time step)

$$\mathbf{x}(t+\Delta t) = e^{D\Delta t}\mathbf{x}(t) + e^{D(t+\Delta t)}\int_t^{t+\Delta t} e^{-Ds}F(s)\,ds.$$

Using the trapezoidal approximation we have

$$\mathbf{x}(t+\Delta t) = e^{D\Delta t}\mathbf{x}(t) + (\tfrac{1}{2}\Delta t)\,[F(t+\Delta t) + e^{D\Delta t}F(t)].$$

Since D and $F(t+\Delta t)$ are not known explicitly, the above formula provides the basis for an iterative procedure. The error estimate for our integration scheme is seen to be of $O(\Delta t^2)$.

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